

## Practice Second Midterm Exam IV

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This exam is closed-book and closed-computer. You may have a double-sided, 8.5" × 11" sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 36 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

Question	Points
(1) Graphs and the Pigeonhole Principle	/ 8
(2) Induction	/ 10
(3) Binary Relations	/ 8
(4) Functions	/ 10
	<b>/ 36</b>

**Problem One: Graphs and the Pigeonhole Principle****(8 Points)***(Midterm Exam, Winter 2016)*

On Problem Set Four, you proved that a graph with  $n^2+1$  nodes must either have a chromatic number of at least  $n+1$  or an independent set of size at least  $n+1$ . In this problem, you'll prove an analogous result, but this time focusing on *edges* rather than the *nodes*.

Let's begin with some new definitions. First, we'll say that a **matching** in a graph  $G = (V, E)$  is a set  $M \subseteq E$  of edges in  $G$  such that no two edges in  $M$  share an endpoint. The **size** of a matching is the number of edges it contains. The **matching number** of a graph  $G$ , denoted  $\nu(G)$ , is the size of the largest matching in  $G$ .

Now, let's introduce a variation on a definition we've seen before. A  **$k$ -edge coloring** of a graph  $G = (V, E)$  is a way of coloring each of the edges in  $G$  one of  $k$  different colors so that no two edges that share an endpoint are assigned the same color. The **chromatic index** of a graph  $G$ , denoted  $\chi_1(G)$ , is the minimum number of colors needed in any edge coloring of  $G$ .

Let  $G$  be an undirected graph with exactly  $n^2+1$  edges for some natural number  $n \geq 1$ . Prove that either  $\chi_1(G) \geq n+1$  or  $\nu(G) \geq n+1$  (or both).

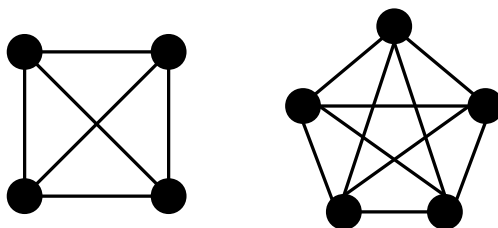
*(Extra space for your answer to Problem One, if you need it.)*

**Problem Two: Induction****(10 Points)**

(A classic practice exam question we've been using for years)

On Problem Set Four and Problem Set Five, you learned how to reason about graphs and mathematical induction. In this problem, you'll see a new type of graph called the ***k*-clique**, then will use induction to prove a useful property of *k*-cliques with applications to social network analysis.

A ***k*-clique** is a graph with *k* nodes where each node is connected to the *k*-1 other nodes in the graph. For example, here's a 4-clique and a 5-clique:



Now, suppose that you take a *k*-clique and color each edge either red or blue. Prove the following result by induction: if the *k*-clique contains an odd-length simple cycle made only of blue edges, then it must contain a simple cycle of length three with an odd number of blue edges (that is, a simple cycle of length three with exactly one blue edge or exactly three blue edges.) This result might seem pretty strange, but trust me, it's meaningful. We'll put details in the solution set. ☺

As a hint, try doing induction on the length of the cycle rather than the number of nodes in the graph.

*(Extra space for your answer to Problem Two, if you need it)*

**Problem Three: Binary Relations****(8 Points)***(CS103 Midterm, Fall 2015)*

In this question, we're going to introduce a few new definitions pertaining to binary relations, then ask you to play around with these definitions and see how they relate to concepts you've seen in lecture and on Problem Set Three.

Let's begin with a new definition. We'll say that a binary relation  $R$  over a set  $A$  is called ***antitransitive*** if the following statement is true:

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \wedge bRc \rightarrow \neg aRc)$$

Next, we'll say that a binary relation  $R$  over a set  $A$  is called a ***nonequivalence relation*** if it is irreflexive, symmetric, and antitransitive.

Finally, if  $R$  is a binary relation over a set  $A$ , then we'll say that the ***complement of  $R$*** , denoted  $\bar{R}$ , is a binary relation over  $A$  defined as follows:

$$a\bar{R}b \text{ if } \neg aRb$$

Prove that if  $R$  is an equivalence relation over  $A$ , then  $\bar{R}$  is a nonequivalence relation over  $A$ .

*(Extra space for your answer to Problem Three, if you need it.)*

**Problem Four: Functions****(10 Points)***(CS103 Midterm, Winter 2016)*

Throughout the quarter, we've discussed the importance of writing proofs that call back to formal definitions. As we've explored different concepts in discrete mathematics (functions, relations, cardinality, graphs, etc.), we've provided formal definitions for the terms at hand and then written proofs based on those definitions. You've gotten a lot of practice with this throughout the problem sets. In this problem, we're going to introduce a new definition, then ask you to prove properties about how this new definition relates to the definitions we've seen over the course of the quarter.

Let's begin with a refresher on one of the lesser-used set operations. If  $S$  and  $T$  are sets, then  $S - T$  is the set of all elements  $x$  where  $x \in S$  but  $x \notin T$ . Specifically,  $S - T = \{ x \mid x \in S \wedge x \notin T \}$ .

Now, let's introduce some new terminology. Let  $f : A \rightarrow B$  be any function and let  $S$  be a subset of  $A$ . The **image of  $S$  under  $f$** , denoted  $f[S]$ , is the set of values produced by applying  $f$  to each element of  $S$ . Formally:

$$f[S] = \{ b \in B \mid \text{there is some } a \in S \text{ such that } f(a) = b \}$$

Here are some examples:

- If  $f : \mathbb{N} \rightarrow \mathbb{N}$  is the function  $f(n) = n + 2$ , then  $f[\{1, 2, 3\}] = \{3, 4, 5\}$  because  $f(1) = 3$ ,  $f(2) = 4$ , and  $f(3) = 5$ .
- If  $g : \mathbb{Z} \rightarrow \mathbb{N}$  is the function  $g(x) = x^2$ , then  $g[\{-1, 0, 1, 2\}] = \{0, 1, 4\}$  because  $g(-1) = 1$ ,  $g(0) = 0$ ,  $g(1) = 1$ , and  $g(2) = 4$ .
- If  $h : \mathbb{N} \rightarrow \mathbb{N}$  is the function  $h(n) = 103$ , then  $h[\emptyset] = \emptyset$  because there are no elements in  $\emptyset$  to which we can apply  $f$ .

Your task is to prove the following result: if  $f : A \rightarrow B$  is an arbitrary **bijection** and  $S \subseteq A$  is an arbitrary subset of  $A$ , then  $f[A - S] = B - f[S]$ . We've broken this down into two steps.

- (5 Points)** Prove that  $B - f[S] \subseteq f[A - S]$ .



*(Extra space for your answer to Problem Four, Part (i), if you need it.)*

ii. **(5 Points)** Prove that  $f[A - S] \subseteq B - f[S]$